Adams Spectral Sequence

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Outline

Multiplicative structure of Adams SS

Definition

M,N,P are R-modules, C_*,D_* are projective resolution of M,N. For $[\alpha] \in Ext_R^{s,t}(M,N)$, $[\beta] \in Ext_R^{u,v}(N,P)$, we can represent them by $C_s \to \Sigma^t N$,$D_u \to \Sigma^v P$. Then we can define $[\beta][\alpha] \in Ext_R^{s+u,t+v}(M,P)$ by $\Sigma^t \beta \circ \alpha : C_{s+u} \to \Sigma^{t+v} P$. It can be represent by the following diagram:

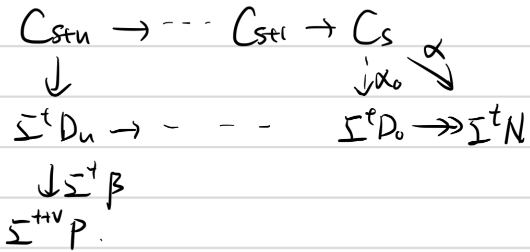


Figure: multi. diagram

Multiplicative structure of Adams SS

Theorem

This multiplicative satisfy:

- $d_r(\alpha\beta) = d_r(\alpha)\beta + (-1)^{s+u}\alpha d_r(\beta)$
- The multiplicative structure on E_{r+1} is induced by that on E_r
- The multiplicative structure on E_∞ corresponds to the multiplicative structure on $\pi_*(X)$ (need X is a ring spectrum)

Proof

Ref to Section 2.3 of Green Book
(Basic idea: Consider the product of two (minimal) resolution.

Calculation of E_2 page

To calculate the E_2 page of (classical) Adams SS, We have the following methods:

- Minimal resolution.
- May SS
- Lambda algebra

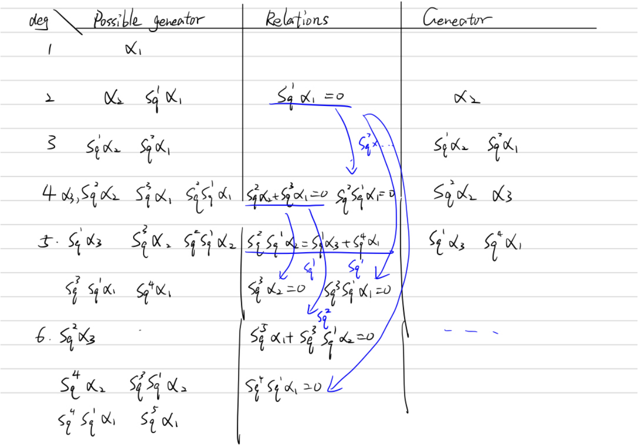
Minimal resolution

Steps of calculating minimal resolution: Additive structure

$$\ldots\ldots \to B_2 \to B_1 \to B_0 = A_2 \to \mathbb{Z}_2 \tag{1}$$

- Find kernel of $A_2 \to \mathbb{Z}_2$:$\alpha_k = Sq^{2^{k-1}}$
- Generate B_1 by α_i freely (as $A_2 mod$)
- Find $ker(B_1 \to B_0)$: the generator of "relation"

Minimal resolution



deg	Possible generator	Relations	Generator
1	α_1		
2	α_2 $Sq^1\alpha_1$	$Sq^1\alpha_1 = 0$	α_2
3	$Sq^1\alpha_2$ $Sq^2\alpha_1$		$Sq^1\alpha_2$ $Sq^2\alpha_1$
4	$\alpha_3, Sq^2\alpha_2$ $Sq^3\alpha_1$ $Sq^2Sq^1\alpha_1$	$Sq^2\alpha_2+Sq^3\alpha_1=0$ $Sq^2Sq^1\alpha_1=0$	$Sq^2\alpha_2$ α_3
5.	$Sq^1\alpha_3$ $Sq^3\alpha_2$ $Sq^2Sq^1\alpha_2$ $Sq^3Sq^1\alpha_1$ $Sq^4\alpha_1$	$Sq^2Sq^1\alpha_2 = Sq^1\alpha_3+Sq^4\alpha_1$ $Sq^3\alpha_2=0$ $Sq^3Sq^1\alpha_1=0$	$Sq^1\alpha_3$ $Sq^4\alpha_1$
6.	$Sq^2\alpha_3$ $Sq^4\alpha_2$ $Sq^3Sq^1\alpha_2$ $Sq^4Sq^1\alpha_1$ $Sq^5\alpha_1$	$Sq^5\alpha_1+Sq^3Sq^1\alpha_2=0$ $Sq^4Sq^1\alpha_1=0$	- - -

Figure: Table of generators & relations

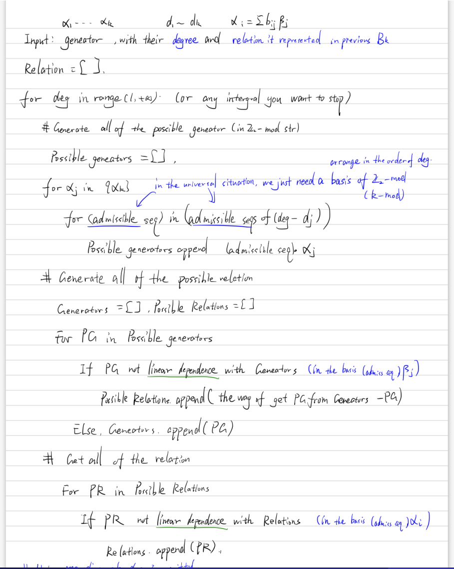
Minimal resolution

Deg	Generators	Relations
2	β_1	
3		$\underline{Sq^1 \beta_1 = 0}$
4	β_2, $Sq^2 \beta_1$	
5	β_3, $Sq^3 \beta_1$, $Sq^1 \beta_2$	$Sq^2 Sq^1 \beta_1 = 0$
6	$Sq^4 \beta_1$, $Sq^1 \beta_3$	$\underline{Sq^2 \beta_2 = Sq^4 \beta_1 + Sq^1 \beta_3}$, $Sq^3 Sq^1 \beta_1 = 0$
7	$Sq^5 \beta_1$, $Sq^2 \beta_3$, $Sq^2 Sq^1 \beta_2$	$Sq^3 \beta_2 = Sq^5 \beta_1$, $Sq^4 Sq^1 \beta_1 = 0$
8	β_4, $Sq^6 \beta_1$, $Sq^4 \beta_2$, $Sq^4 Sq^2 \beta_1$, $Sq^3 Sq^1 \beta_2$, $Sq^3 \beta_3$	$Sq^2 Sq^1 \beta_3 = Sq^3 Sq^1 \beta_2 + Sq^6 \beta_1$, $Sq^5 Sq^1 \beta_1 = 0$
9	β_5, $Sq^7 \beta_1$, $Sq^5 Sq^2 \beta_1$, $Sq^5 \beta_2$, $Sq^2 \beta_4$,	$Sq^7 \beta_1 = Sq^3 Sq^1 \beta_3$, $Sq^5 \beta_2 = Sq^4 Sq^1 \beta_2$, $Sq^4 Sq^2 Sq^1 \beta_1 = 0$, $Sq^6 Sq^1 \beta_1 = 0$

Figure: $ker(B_1 \to B_0)$, low deg part

Minimal resolution

We can write these steps as pseudocode:



$\alpha_1 \cdots \alpha_k$ $d_1 \sim d_k$ $\alpha_i = \sum b_{ij} \beta_j$

Input: generator, with their degree and relation it represented in previous B_k

Relation = [ ].

for deg in range(1, +∞): (or any integral you want to stop)

  # Generate all of the possible generator (in $\mathbb{Z}_2$-mod str)

  Possible generators = [ ], arrange in the order of deg.

  for α_j in $\{\alpha_k\}$ (in the universal situation, we just need a basis of $\mathbb{Z}_2$-mod (k-mod))

    for (admissible seq) in (admissible seqs of (deg − d_j))

      Possible generators append (admissible seq)·α_j

  # Generate all of the possible relation

  Generators = [ ], Possible Relations = [ ]

  For PG in Possible generators

    If PG not linear dependence with Generators (in the basis (admiss seq) β_j)

      Possible Relations. append(the way of get PG from Generators − PG)

    Else, Generators. append(PG)

  # Get all of the relation

  For PR in Possible Relations

    If PR not linear dependence with Relations (in the basis (admiss seq) α_i)

      Relations. append (PR).

Minimal resolution

Steps of calculating minimal resolution: Multi. structure:
Follow the definition of multi. structure, we get a diagram:

$$
\begin{aligned}
\beta_1 &\mapsto \mathrm{Sq}^1 \alpha_1, \\
\beta_2 &\mapsto \mathrm{Sq}^2 \alpha_2 + \mathrm{Sq}^3 \alpha_1, \\
\beta_3 &\mapsto \mathrm{Sq}^2 \mathrm{Sq}^1 \alpha_2 + \mathrm{Sq}^1 \alpha_3 + \mathrm{Sq}^4 \alpha_1, \\
\beta_4 &\mapsto \mathrm{Sq}^4 \alpha_3 + \mathrm{Sq}^7 \alpha_1 + \mathrm{Sq}^6 \alpha_2, \\
\beta_5 &\mapsto \mathrm{Sq}^4 \mathrm{Sq}^1 \alpha_3 + \mathrm{Sq}^1 \alpha_4 + \mathrm{Sq}^8 \alpha_1 + \mathrm{Sq}^7 \alpha_2, \\
\beta_6 &\mapsto \mathrm{Sq}^2 \alpha_4 + \mathrm{Sq}^4 \mathrm{Sq}^2 \alpha_3 + \mathrm{Sq}^8 \alpha_2 + \mathrm{Sq}^7 \mathrm{Sq}^2 \alpha_1.
\end{aligned}
$$

Figure: multi. diagram

Minimal resolution

$f_1\colon B_1 \to B_2$	$f_2\colon B_2 \to B_3$	$f_3\colon B_3 \to B_4$	$f_4\colon B_4 \to B_5$
$\alpha_1 \mapsto 1$	$\beta_1 \mapsto \alpha_1$	$\gamma_1 \mapsto \beta_1$	$\delta_1 \mapsto \gamma_1$
$\alpha_2 \mapsto 0$	$\beta_2 \mapsto \mathrm{Sq}^1\,\alpha_2$	$\gamma_2 \mapsto \beta_2$	$\delta_2 \mapsto \gamma_3$
$\alpha_3 \mapsto 0$	$\beta_3 \mapsto \alpha_3$	$\gamma_3 \mapsto \beta_5 + \mathrm{Sq}^1\,\beta^4$	
$\alpha_4 \mapsto 0$	$\beta_4 \mapsto \mathrm{Sq}^3\,\alpha_3$	$\gamma_4 \mapsto \mathrm{Sq}^1\,\beta_5 + \mathrm{Sq}^2\,\beta_4 + \mathrm{Sq}^8\,\beta_1$	
$\alpha_5 \mapsto 0$	$\beta_5 \mapsto \alpha_4$	$\gamma_5 \mapsto \mathrm{Sq}^1\,\beta_6$	
	$\beta_6 \mapsto \mathrm{Sq}^7\,\alpha_2$		

Figure: Table of multi. $\alpha_1(h_0)$

Minimal resolution

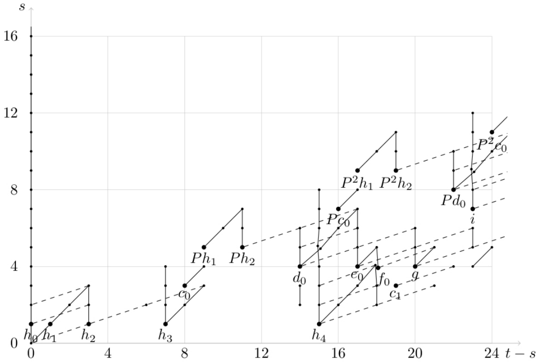


FIGURE 1.9. Indecomposables in $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \le t - s \le 24$

Figure: E_2 page of Adams SS, $0 \le t - s \le 24$

Minimal resolution

What's more, we can define Steenrod operation

$$Sq^i : Ext_A^{s,t}(H^*(X), \mathbb{Z}) \to Ext_A^{s+i,2t}(H^*(X), \mathbb{Z}) \tag{2}$$

on E_2 page. It can be defined as follow:

A little more generally, there are Steenrod operations

$$Sq^i\colon \operatorname{Ext}_\Gamma^{s,t}(L, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_\Gamma^{s+i,2t}(L, \mathbb{F}_2)$$

for any cocommutative Γ-module coalgebra L. Let $C_* \to L$ be a free Γ-module resolution, and let $\Delta\colon W_* \otimes C_* \to C_* \otimes C_*$ be a Σ_2-equivariant map of Γ-module complexes covering the coproduct $\psi\colon L \to L \otimes L$. For each cocycle $x\colon C_s \to \Sigma^t\mathbb{F}_2$ the composite

$$C_{2s-i} \cong \mathbb{F}_2\{e_i\} \otimes C_{2s-i} \subset W_i \otimes C_{2s-i} \subset (W_* \otimes C_*)_{2s} \xrightarrow{\Delta} (C_* \otimes C_*)_{2s} \xrightarrow{x \otimes x} \Sigma^t\mathbb{F}_2 \otimes \Sigma^t\mathbb{F}_2 \cong \Sigma^{2t}\mathbb{F}_2$$

Figure: Definition of Steenrod operation on Ext

Minimal resolution

if X is a H_∞ ring spectrum. Then there is a relation between differential and Steenrod operation:

DEFINITION 11.21. Let $A \in E_2^{s,t}$, $B_1 \in E_2^{s+r_1,t+r_1-1}$ and $B_2 \in E_2^{s+r_2,t+r_2-1}$ be classes in a spectral sequence with differentials $d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. The notation

$$d_*(A) = B_1 \dotplus B_2$$

means that $d_r(A) = 0$ for $2 \le r < \min\{r_1, r_2\}$, while

$$\begin{cases} d_{r_1}(A) = B_1 & \text{if } r_1 < r_2, \\ d_r(A) = B_1 + B_2 & \text{if } r_1 = r = r_2, \text{ and} \\ d_{r_2}(A) = B_2 & \text{if } r_1 > r_2. \end{cases}$$

THEOREM 11.22 ([**45**, Thm. VI.1.1 and VI.1.2]). *Let $E_r^{*,*}(Y)$ be the mod 2 Adams spectral sequence for an H_∞ ring spectrum Y, and let $x \in E_2^{s,t}(Y)$ be an element that survives to the E_r-term, where $r \ge 2$. Let $0 \le i \le s$, and let $v = v(t-i)$, $a = a(t-i)$ and $\bar{a}$ be as just defined. Then*

$$d_*(Sq^i(x)) = Sq^{i+r-1}(d_r(x)) \dotplus \begin{cases} 0 & \text{if } v > s-i+1, \\ \bar{a}\, x\, d_r(x) & \text{if } v = s-i+1, \\ \bar{a}\, Sq^{i+v}(x) & \text{if } v = s-i \text{ or } v \le \min\{s-i, 10\}. \end{cases}$$

Figure: Relation between Steenrod operation and differential

Basic information

Prop

$$Ext_A^{s,t}(H^*(X), \mathbb{Z}) \cong Ext_{A_*}^{s+i,2t}(\mathbb{Z}, H_*(X)) \tag{3}$$

Prop

$A_* = P(\xi_i, \xi_2, \dots)$ where $|\xi_n| = 2^n - 1$, and the coproduct on A_* is given by

$$\Delta \xi_n = \sum_{0 \le i \le n} \xi_{n-i}^{2^i} \otimes \xi_i \tag{4}$$

Basic information

To calculate this Ext, we can construct cobar complex as the A_* injective comodule resolution:

3.1.2. Proposition. *The E_2-term for the classical Adams spectral sequence for $\pi_*(X)$ is the cohomology of the cobar complex $C^*_{A_*}(H_*(X))$ defined by*

$$C^s_{A_*}(H_*(X)) = \bar{A}_* \otimes \cdots \otimes \bar{A}_* \otimes H_*(X)$$

(with s tensor factors of $\bar{A}_*$). For $a_i \in A_*$ and $x \in H_*(X)$, the element $a_1 \otimes \cdots a_s \otimes x$ will be denoted by $[a_1|a_2|\cdots|a_s]x$. The coboundary operator $d_s \colon C^s_{A_*}(H_*(X)) \to C^{s+1}_{A_*}(H_*(X))$ is given by

$$d_s[a_1|\cdots|a_s]x = [1|a_1|\cdots|a_s]x + \sum_{i=1}^{s}(-1)^i[a_1|\cdots|a_{i-1}|a_i'|a_i''|a_{i+1}|\cdots|a_s]x + (-1)^{s+1}[a_1|\cdots|a_s|x']x'',$$

where $\Delta a_i = a_i' \otimes a_i''$ and $\psi(x) = x' \otimes x'' \in A_* \otimes H_*(X)$. [A priori this expression lies in $A_*^{\otimes s+1} \otimes H_*(X)$. The diligent reader can verify that it actually lies in $\bar{A}_*^{\otimes s+1} \otimes H_*(X)$.] □

Figure: Definition of cobar complex, and its relation to Adams SS

May SS

Definition

For p=2,

$$E^0 A_* = E(\xi_{i,j} : i > 0, j \geq 0) \tag{5}$$

with coproduct

$$\Delta \xi_{i,j} =_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j} \tag{6}$$

where $\xi_{0,j} = 1$ and $\xi_{i,j}$ is the projection of $\xi_i^{2^j}$

Theorem

For p=2, $Ext^{***}_{E^0 A_*}(\mathbb{Z}_2, \mathbb{Z}_2)$ is the cohomology of the complex

$$V^{***} = P(h_{i,j} : i > 0, j \geq 0) \tag{7}$$

with $d_{i,j} = \sum_{0<k<i} h_{k,j} h_{i-k,j+k}$, where $h_{i,j} \in V^{1,2^j(2^i-1),i}$ corresponds to $\xi_{i,j} \in A_*$

May SS

Theorem

There is a spectral sequence converging to

$$Ext_{A_*}^{**}(\mathbb{Z}_2, \mathbb{Z}_2) \tag{8}$$

with $V^{***} = E_1^{***}$ and $d_r : E_r^{s,t,u} \to E_r^{s+1,t,u+1-r}$

Pf: Green book

May SS

3.2.8. LEMMA. *In the range* $t-s \leq 13$ *the* E_2*-term for the May spectral sequence* (3.2.3) *has generators*

$$h_j = h_{1,j} \in E_2^{1,2^j,1},$$
$$b_{i,j} = h_{i,j}^2 \in E_2^{2,2^{j+1}(2^i-1),2i},$$

and

$$x_7 = h_{20}h_{21} + h_{11}h_{30} \in E_2^{2,9,4}$$

with relations

$$h_j h_{j+1} = 0,$$
$$h_2 b_{20} = h_0 x_7,$$

and

$$h_2 x_7 = h_0 b_{21}.$$ □

Figure: E_2 page of May SS

lambda algebra

More precisely, Λ is a bigraded $\mathbf{Z}/(2)$-algebra with generators $\lambda_n \in \Lambda^{1,n+1}$ $(n \geq 0)$ and relations

$$\lambda_i \lambda_{2i+1+n} = \sum_{j \geq 0} \binom{n-j-1}{j} \lambda_{i+n-j} \lambda_{2i+1+j} \quad \text{for } i, n \geq 0 \tag{3.3.1}$$

with differential

$$d(\lambda_n) = \sum_{j \geq 1} \binom{n-j}{j} \lambda_{n-j} \lambda_{j-1}. \tag{3.3.2}$$

Note that d behaves formally like left multiplication by λ_{-1}.

3.3.3. DEFINITION. *A monomial $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_s} \in \Lambda$ is admissible if $2i_r \geq i_{r+1}$ for $1 \leq r < s$. $\Lambda(n) \subset \Lambda$ is the subcomplex spanned by the* admissible *monomials with $i_1 < n$.*

The following is an easy consequence of 3.3.1 and 3.3.2.

3.3.4. PROPOSITION.
(a) *The admissible monomials constitute an additive basis for Λ.*
(b) *There are short exact sequences of complexes*

$$0 \to \Lambda(n) \to \Lambda(n+1) \to \Sigma^n \Lambda(2n+1) \to 0. \qquad \square$$

The significant property of Λ is the following.

3.3.5. THEOREM (Bousfield *et al.* [2]). (a) $H(\Lambda) = \mathrm{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$, *the classical Adams E_2-term for the sphere.*

(b) *$H(\Lambda(n))$ is the E_2-term of a spectral sequence converging to $\pi_*(S^n)$.*

(c) *The long exact sequence in cohomology* (3.3.6) *given by* 3.3.4(b) *corresponds to the EHP sequence, i.e., to the long exact sequence of homotopy groups of the fiber sequence (at the prime 2)*

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1} \quad \text{(see 1.5.1).} \qquad \square$$

Figure: Def. of lambda algebra

Davis Mahowald SS

Proposition 2.3. *Suppose that we have chosen a sequence of* Γ*-modules* N_σ*, for* $\sigma \geq 0$*, and an exact chain complex*

$$\ldots \xrightarrow{\partial_3} \Gamma//\Lambda \otimes N_2 \xrightarrow{\partial_2} \Gamma//\Lambda \otimes N_1 \xrightarrow{\partial_1} \Gamma//\Lambda \otimes N_0 \xrightarrow{\epsilon} k \to 0$$

of Γ*-modules with diagonal* Γ*-action. Then there is a strongly convergent trigraded spectral sequence*

$$E_1^{\sigma,s,t} = \operatorname{Ext}_\Lambda^{s-\sigma,t}(N_\sigma \otimes M, k) \Longrightarrow_\sigma \operatorname{Ext}_\Gamma^{s,t}(M,k)\,.$$

The d_r*-differentials have* (σ, s, t)*-tridegree* $(r, 1, 0)$ *and there are isomorphisms*

$$E_\infty^{\sigma,s,t} \cong F^\sigma \operatorname{Ext}^{s,t}(M)/F^{\sigma+1}\operatorname{Ext}^{s,t}(M)$$

for all σ*,* s *and* t*, where* $\{F^\sigma \operatorname{Ext}^{s,t}(M)\}_\sigma$ *is a finite and exhaustive filtration of* $\operatorname{Ext}^{s,t}(M) = \operatorname{Ext}_\Gamma^{s,t}(M,k)$.

Figure: Davis Mahowald SS

Calculating the differential

- Target=0
- $d_r \circ d_r = 0$
- $d_r(xy) = d_r(x)y + xd_r(y)$
- Hurewicz map(e.g. $S \to tmf$)
- Steenrod operation

Calculating the differential

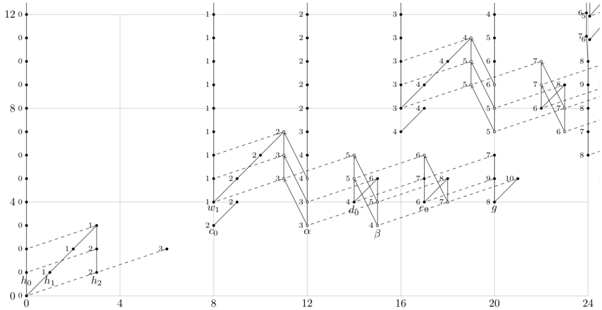


Figure: E_2 page of tmf

The End