Spectral sequence calculation for unstable v_n-periodic homotopy groups of spheres

Zhonglin Wu

Southern Univ. of Science and Technology (SUSTech)

October 10, 2023

Outline

Outline

v_n-periodic homotopy group in stable range

Definition

The v_n-homotopy group (with coefficients in V) for a spectrum Z and a spectrum V which supports a v_n-self-map v:

$$v_n^{-1}\pi_*(Z;V) := v^{-1}[\Sigma^* V, Z]_{Sp}. \tag{1}$$

v_n-periodic homotopy group in stable range

Definition

The v_n-homotopy group (with coefficients in V) for a spectrum Z and a spectrum V which supports a v_n-self-map v:

$$v_n^{-1}\pi_*(Z;V) := v^{-1}[\Sigma^* V, Z]_{Sp}. \tag{1}$$

Theorem

For

$$T(n) = v_n^{-1}V := hocolim(V \xrightarrow{v} \Sigma^{-k}V \xrightarrow{v} \Sigma^{-2k}V \xrightarrow{v} \cdots), \tag{2}$$

$v_n^{-1}\pi_*$*-isomorphism is equivalent to* $T(n)_*$*-isomorphism.*

v_n-periodic homotopy group in stable range

- $L_n^f Sp$: the category of $\bigoplus_{i=0}^{n} T(i)_*$-local spectra.

v_n-periodic homotopy group in stable range

- $L_n^f Sp$: the category of $\bigoplus_{i=0}^{n} T(i)_*$-local spectra.
- $T(n)$-version chromatic tower: $\cdots \to L_2^f Z \to L_1^f Z \to L_0^f Z$.

v_n-periodic homotopy group in stable range

- $L_n^f Sp$: the category of $\bigoplus_{i=0}^n T(i)_*$-local spectra.
- $T(n)$-version chromatic tower:$\cdots \to L_2^f Z \to L_1^f Z \to L_0^f Z$.
- $M_n^f Z := fib(L_n^f Z \to L_{n-1}^f Z)$.

v_n-periodic homotopy group in stable range

- $L_n^f Sp$: the category of $\bigoplus_{i=0}^n T(i)_*$-local spectra.
- $T(n)$-version chromatic tower:$\cdots \to L_2^f Z \to L_1^f Z \to L_0^f Z$.
- $M_n^f Z := fib(L_n^f Z \to L_{n-1}^f Z)$.
- $L_{T(n)} : Ho(M_n^f Sp) \leftrightarrows Ho(Sp_{T(n)}) : M_n^f$ gives equivalence.

v_n-periodic homotopy group in stable range

- $L_n^f Sp$: the category of $\bigoplus_{i=0}^n T(i)_*$-local spectra.
- $T(n)$-version chromatic tower: $\cdots \to L_2^f Z \to L_1^f Z \to L_0^f Z$.
- $M_n^f Z := fib(L_n^f Z \to L_{n-1}^f Z)$.
- $L_{T(n)} : Ho(M_n^f Sp) \leftrightarrows Ho(Sp_{T(n)}) : M_n^f$ gives equivalence.
- $v_n^{-1} V \simeq M_n^f V \simeq L_{T(n)} V$.

v_n-periodic homotopy group in stable range

- $L_n^f Sp$: the category of $\bigoplus_{i=0}^n T(i)_*$-local spectra.
- $T(n)$-version chromatic tower: $\cdots \to L_2^f Z \to L_1^f Z \to L_0^f Z$.
- $M_n^f Z := fib(L_n^f Z \to L_{n-1}^f Z)$.
- $L_{T(n)} : Ho(M_n^f Sp) \leftrightarrows Ho(Sp_{T(n)}) : M_n^f$ gives equivalence.
- $v_n^{-1} V \simeq M_n^f V \simeq L_{T(n)} V$.
- $v_n^{-1}\pi_*(Z; V) \cong [\Sigma^* L_{T(n)} V, L_{T(n)} Z]_{Sp}$.

Unstable v_n-periodic homotopy group

Idea: use above definition on pointed space.

Definition

If a finite type n complex V admits a v_n-self-map:

$$v : \Sigma^{k(N_0+1)}V \to \Sigma^{kN_0}V \tag{3}$$

for some $N_0 \gg 0$. For any $X \in Top_*$, the unstable v_n-periodic homotopy group can be defined as:

$$v_n^{-1}\pi_*(X; V) := v^{-1}[\Sigma^* V, X]_{Top_*} \tag{4}$$

for $n > 0$.

Unstable v_n-periodic homotopy group

- $T(n)$-version chromatic tower of space: $\cdots \to L_2^f X \to L_1^f X \to L_0^f X$, with similar $M_n^f X$ exist.

Unstable v_n-periodic homotopy group

- $T(n)$-version chromatic tower of space:$\cdots \to L_2^f X \to L_1^f X \to L_0^f X$, with similar $M_n^f X$ exist.
- $v_n^{-1}\pi_*(X;V) \cong [\Sigma^* V, M_n^f X]_{Top_*}$.

Φ_V's definition

Idea: try to use a functor to pull above homotopy group back to stable range:

Definition

For (V, v), a finite type n complex with self-map v, $\Phi_V(X)$ is a t-periodic spectrum whose 0^{th} space is given by the direct limit of the sequence:

$$Map_*(V, X) \to Map_*(\Sigma^t V, X) \to Map_*(\Sigma^{2t} V, X) \to \cdots . \quad (5)$$

We have $\pi_*(\Phi_V(X)) \cong v_n^{-1}\pi_*(X; V)$.

Φ_V to Φ_n

Idea: to get rid of (V, v), we need some universal object.
Take a suitable inverse system V_i of finite type n spectra so that

$$holim v_n^{-1} V_i \simeq L_{T(n)} S^0. \tag{6}$$

Φ_n is defined as:

$$\Phi_n(X) := holim \Phi_{V_i}(X). \tag{7}$$

(Completed) unstable v_n-homotopy group (without coefficients) is defined as:

$$v_n^{-1}\pi_*(X) := \pi_*\Phi_n(X). \tag{8}$$

Properties of Φ_n

This functor is unique and satisfies the following properties.

- Φ_n preserves fiber sequences.

Properties of Φ_n

This functor is unique and satisfies the following properties.

- Φ_n preserves fiber sequences.
- $v_n^{-1}\pi_*(X;V) = [\Sigma^* V, \Phi_n(X)]_{Sp}$.

Properties of Φ_n

This functor is unique and satisfies the following properties.

- Φ_n preserves fiber sequences.
- $v_n^{-1}\pi_*(X;V) = [\Sigma^* V, \Phi_n(X)]_{Sp}$.
- If Z is a spectrum, then $\Phi_n \Omega^\infty Z = L_{T(n)} Z$.

Stable v_n homotopy vs unstable case

The stable v_n-periodic homotopy category admits a fully faithful embedding into the unstable v_n-periodic homotopy category:

$$Ho(Sp_{T(n)}) \xrightarrow{(\Omega^\infty M_n^f -)^{\geq d_n}} Ho(M_n^f Top_*) \xrightarrow{\Phi_n} Ho(Sp_{T(n)}). \quad (9)$$

In other words, they are compatible.

$\Phi_{K(n)}$

- The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$.

$\Phi_{K(n)}$

- The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$.
- $\langle K(n)\rangle \geq \langle T(n)\rangle$.

$\Phi_{K(n)}$

- The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$.
- $\langle K(n)\rangle \geq \langle T(n)\rangle$.
- Can we use $K(n)$ instead of $T(n)$? (Telescope conjecture)

$\Phi_{K(n)}$

- The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$.
- $\langle K(n)\rangle \geq \langle T(n)\rangle$.
- Can we use $K(n)$ instead of $T(n)$? (Telescope conjecture)
- $n = 1$ true, $n \geq 2$ false.

$\Phi_{K(n)}$

- The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$.
- $\langle K(n) \rangle \geq \langle T(n) \rangle$.
- Can we use $K(n)$ instead of $T(n)$? (Telescope conjecture)
- $n = 1$ true, $n \geq 2$ false.
- However, we still use $K(n)$ to compute since we know little about $T(n)$, at least computationally.

Outline

The Goodwillie Tower

- Goodwillie tower: Taylor series of a functor which divides a functor into n-excisive part.

The Goodwillie Tower

- Goodwillie tower: Taylor series of a functor which divides a functor into n-excisive part.
- $\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X)$.

The Goodwillie Tower

- Goodwillie tower: Taylor series of a functor which divides a functor into n-excisive part.
- $\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X)$.
- $D_n(X) := fib(P_n(X) \to P_{n-1}(X))$.

Properties of the Goodwillie Tower

- $P_k\Phi_n \simeq \Phi_n P_k Id$.

Properties of the Goodwillie Tower

- $P_k\Phi_n \simeq \Phi_n P_k Id$.
- $D_n(X) = \Omega^\infty((\Sigma^\infty X)^n \wedge \mathcal{O}(n))_{h\Sigma_n}$.

Properties of the Goodwillie Tower

- $P_k\Phi_n \simeq \Phi_n P_k Id$.
- $D_n(X) = \Omega^\infty((\Sigma^\infty X)^n \wedge \mathcal{O}(n))_{h\Sigma_n}$.
- $D_{p^k}(\Phi_{K(n)}(S^q)) \simeq \Omega^\infty \Sigma^{q-k} L(k)_q$ and $D_i(\Phi_{K(n)}(S^q)) \simeq *$ for q odd.

Wang's approach

- **Idea: calculate $\Phi_{K(n)}(X)$ by a SS induced by GT, then use it in ANSS(or sth else).**

Wang's approach

- **Idea: calculate $\Phi_{K(n)}(X)$ by a SS induced by GT, then use it in ANSS(or sth else).**
- $E_{n*}(D_k(\Phi_{K(n)}(S^q))) \xrightarrow{GT} E_{n*}(\Phi_{K(n)}(S^q)) \xrightarrow{ANSS} \pi_*(\Phi_{K(n)}(S^q))$.

Rezk's approach

- **Calculate $E_{n*}(\Phi_{K(n)}(S^q))$ in another way?**

Rezk's approach

- **Calculate $E_{n*}(\Phi_{K(n)}(S^q))$ in another way?**
-
$$\Phi_{K(n)}(X) \xrightarrow{C^{S_K}} TAQ_{S_K}(S_K^{X_+}) \xleftarrow{model} \Delta^q. \qquad (10)$$

Rezk's approach

- **Calculate $E_{n*}(\Phi_{K(n)}(S^q))$ in another way?**
-
$$\Phi_{K(n)}(X) \xrightarrow{C^{S_K}} TAQ_{S_K}(S_K^{X_+}) \xleftarrow{model} \Delta^q. \qquad (10)$$
- topological Andre-Quillen cohomology of the augmented commutative S_K-algebra S_K^{X+}.

Rezk's approach

- **Calculate $E_{n*}(\Phi_{K(n)}(S^q))$ in another way?**
- $$\Phi_{K(n)}(X) \xrightarrow{C^{S_K}} TAQ_{S_K}(S_K^{X_+}) \xleftarrow{model} \Delta^q. \qquad (10)$$
- topological Andre-Quillen cohomology of the augmented commutative S_K-algebra S_K^{X+}.
- Δ^q: Dyer-Lashof algebra.

Rezk's approach

- The E_{n*}-homology of the spectrum $L(k)_q$ is isomorphic to the dual of k^{th} term of the Koszul resolution for Δ^q.

Rezk's approach

- The E_{n*}-homology of the spectrum $L(k)_q$ is isomorphic to the dual of k^{th} term of the Koszul resolution for Δ^q.
- $Ext^s_{\Delta^q}(\tilde{E}_{n,t}(S^q), \tilde{E}_{n,t}) \Rightarrow E_{n,q+t-s}(\Phi_{K(n)}(S^q))$.

Summary of the above approach

This approach can be explained by the following diagram:

$$GT \xrightarrow{(-)^{GL_n(\mathbb{F}_p)}} H_c^*(G_n, E_{n*}(\Phi_{K(n)}(S^q))) \xrightarrow{(-)^{D^\times}} \pi_*(\Phi_{K(n)}(S^q)). \tag{11}$$

The problem we are facing

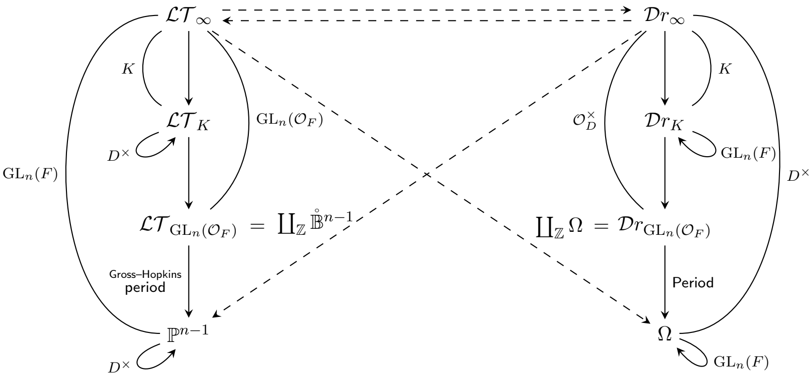
- $\Phi_{K(n)}(S^q)$ is not a ring spectrum. We have few tools to deal with the differential in the E_2-page of the ANSS.

The problem we are facing

- $\Phi_{K(n)}(S^q)$ is not a ring spectrum. We have few tools to deal with the differential in the E_2-page of the ANSS.
- **Idea: swap the order of two SS?**

Outline

New approach



Plan of work

- Understand the origin frame. ($L(n)$, TAQ, Δ^q and so on.)

Plan of work

- Understand the origin frame. ($L(n)$, TAQ, Δ^q and so on.)
- Understand the duality in AG.

Plan of work

- Understand the origin frame. ($L(n)$, TAQ, Δ^q and so on.)
- Understand the duality in AG.
- Try to find these duality in the new frame.

Plan of work

- Understand the origin frame. ($L(n)$, TAQ, Δ^q and so on.)
- Understand the duality in AG.
- Try to find these duality in the new frame.
- Sample explicit calculation.

The End