# Spectral sequence calculation for unstable $v_n$ -periodic homotopy groups of spheres

#### Zhonglin Wu

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October 10, 2023



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Formulation of the problems



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- 2 The methods of calculation
- 3 Formulation of the problems



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### Outline

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2 The methods of calculation

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### $v_n$ -periodic homotopy group in stable range

#### Definition

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The  $v_n$ -homotopy group (with coefficients in V) for a spectrum Z and a spectrum V which supports a  $v_n$ -self-map v:

$$v_n^{-1}\pi_*(Z;V) := v^{-1}[\Sigma^*V, Z]_{Sp}.$$
(1)

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(1)

#### Theorem

#### For

$$T(n) = v_n^{-1}V := hocolim(V \xrightarrow{v} \Sigma^{-k}V \xrightarrow{v} \Sigma^{-2k}V \xrightarrow{v} \cdots), \quad (2)$$

 $v_n^{-1}\pi_*$ -isomorphism is equivalent to  $T(n)_*$ -isomorphism.

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### $v_n$ -periodic homotopy group in stable range

•  $L_n^f Sp$ : the category of  $\bigoplus_{i=0}^n T(i)_*$ -local spectra.



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### $v_n$ -periodic homotopy group in stable range

- $L_n^f Sp$ : the category of  $\bigoplus_{i=0}^n T(i)_*$ -local spectra.
- T(n)-version chromatic tower:  $\cdots \rightarrow L_2^f Z \rightarrow L_1^f Z \rightarrow L_0^f Z$ .



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- $\bullet \ M_n^f Z := fib(L_n^f Z \to L_{n-1}^f Z).$
- $L_{T(n)}: Ho(M_n^f Sp) \leftrightarrows Ho(Sp_{T(n)}): M_n^f$  gives equivalence.



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- $v_n^{-1}\pi_*(Z;V) \cong [\Sigma^* L_{T(n)}V, L_{T(n)}Z]_{Sp}.$

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### Unstable $v_n$ -periodic homotopy group

Idea: use above definition on pointed space.

#### Definition

If a finite type n complex V admits a  $v_n$ -self-map:

$$v: \Sigma^{k(N_0+1)}V \to \Sigma^{kN_0}V \tag{3}$$

for some  $N_0 \gg 0$ . For any  $X \in Top_*$ , the unstable  $v_n$ -periodic homotopy group can be defined as:

$$v_n^{-1}\pi_*(X;V) := v^{-1}[\Sigma^*V,X]_{Top_*}$$
(4)

for n > 0.

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### Unstable $v_n$ -periodic homotopy group

• 
$$T(n)$$
-version chromatic tower of  
space: $\cdots \rightarrow L_2^f X \rightarrow L_1^f X \rightarrow L_0^f X$ , with similar  $M_n^f X$  exist.



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### Unstable $v_n$ -periodic homotopy group

$$\begin{array}{l} & T(n) \text{-version chromatic tower of} \\ \text{space:} \cdots \rightarrow L_2^f X \rightarrow L_1^f X \rightarrow L_0^f X \text{, with similar } M_n^f X \text{ exist.} \\ & v_n^{-1} \pi_*(X;V) \cong [\Sigma^* V, M_n^f X]_{Top_*}. \end{array}$$



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### $\Phi_V$ 's definition

Idea: try to use a functor to pull above homotopy group back to stable range:

#### Definition

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For (V, v), a finite type n complex with self-map v,  $\Phi_V(X)$  is a t-periodic spectrum whose  $0^{th}$  space is given by the direct limit of the sequence:

$$Map_*(V,X) \to Map_*(\Sigma^t V,X) \to Map_*(\Sigma^{2t}V,X) \to \cdots$$
 (5)

We have  $\pi_*(\Phi_V(X)) \cong v_n^{-1}\pi_*(X;V)$ .

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### $\Phi_V$ to $\Phi_n$

Idea: to get rid of (V, v), we need some universal object. Take a suitable inverse system  $V_i$  of finite type n spectra so that

$$holimv_n^{-1}V_i \simeq L_{T(n)}S^0.$$
 (6)

 $\Phi_n$  is defined as:

$$\Phi_n(X) := holim\Phi_{V_i}(X). \tag{7}$$

(Completed) unstable  $v_n$ -homotopy group (without coefficients) is defined as:

$$v_n^{-1}\pi_*(X) := \pi_*\Phi_n(X).$$
 (8)

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#### This functor is unique and satisfies the following properties.

•  $\Phi_n$  preserves fiber sequences.



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• 
$$v_n^{-1}\pi_*(X;V) = [\Sigma^*V, \Phi_n(X)]_{Sp}.$$



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This functor is unique and satisfies the following properties.

- $\Phi_n$  preserves fiber sequences.
- $v_n^{-1}\pi_*(X;V) = [\Sigma^*V, \Phi_n(X)]_{Sp}.$
- If Z is a spectrum, then  $\Phi_n \Omega^{\infty} Z = L_{T(n)} Z$ .



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### Stable $v_n$ homotopy vs unstable case

The stable  $v_n$ -periodic homotopy category admits a fully faithful embedding into the unstable  $v_n$ -periodic homotopy category:

$$Ho(Sp_{T(n)}) \xrightarrow{(\Omega^{\infty}M_n^f -)^{\geq d_n}} Ho(M_n^f Top_*) \xrightarrow{\Phi_n} Ho(Sp_{T(n)}).$$
(9)

In other words, they are compatible.



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### • The K(n)-localization of $\Phi_n$ is denoted as $\Phi_{K(n)} := L_{K(n)} \Phi_n$ .



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• The K(n)-localization of  $\Phi_n$  is denoted as  $\Phi_{K(n)} := L_{K(n)} \Phi_n$ . •  $\langle K(n) \rangle \ge \langle T(n) \rangle$ .



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- Can we use K(n) instead of T(n)? (Telescope conjecture)



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- n = 1 true,  $n \ge 2$  false.



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- The K(n)-localization of  $\Phi_n$  is denoted as  $\Phi_{K(n)} := L_{K(n)} \Phi_n$ . •  $\langle K(n) \rangle \ge \langle T(n) \rangle$ .
- Can we use K(n) instead of T(n)? (Telescope conjecture)
- n = 1 true,  $n \ge 2$  false.
- However, we still use K(n) to compute since we know little about T(n), at least computationally.



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Formulation of the problems

### Outline

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### **1** Definition of $v_n$ -periodic homotopy group

#### 2 The methods of calculation

#### 3 Formulation of the problems



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The methods of calculation

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### The Goodwillie Tower

 Goodwillie tower: Taylor series of a functor which divides a functor into *n*-excisive part.



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### The Goodwillie Tower

 Goodwillie tower: Taylor series of a functor which divides a functor into *n*-excisive part.

 $\bullet \cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X).$ 



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### The Goodwillie Tower

- Goodwillie tower: Taylor series of a functor which divides a functor into *n*-excisive part.
- $\bullet \cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X).$
- $\bullet D_n(X) := fib(P_n(X) \to P_{n-1}(X)).$



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### Properties of the Goodwillie Tower

$$P_k \Phi_n \simeq \Phi_n P_k Id.$$

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### Properties of the Goodwillie Tower

$$P_k \Phi_n \simeq \Phi_n P_k Id.$$
$$D_n(X) = \Omega^{\infty} ((\Sigma^{\infty} X)^n \wedge \mathcal{O}(n))_{h \Sigma_n}.$$



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### Properties of the Goodwillie Tower

- $P_k \Phi_n \simeq \Phi_n P_k Id.$
- $D_n(X) = \Omega^{\infty}((\Sigma^{\infty}X)^n \wedge \mathcal{O}(n))_{h\Sigma_n}.$
- $\quad D_{p^k}(\Phi_{K(n)}(S^q)) \simeq \Omega^{\infty} \Sigma^{q-k} L(k)_q \text{ and } D_i(\Phi_{K(n)}(S^q)) \simeq * \text{ for } q \text{ odd.}$



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■ Idea: calculate  $\Phi_{K(n)}(X)$  by a SS induced by GT, then use it in ANSS(or sth else).



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### Wang's approach

- Idea: calculate  $\Phi_{K(n)}(X)$  by a SS induced by GT, then use it in ANSS(or sth else).



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### Rezk's approach

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• Calculate  $E_{n*}(\Phi_{K(n)}(S^q))$  in another way?



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### Rezk's approach

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Calculate 
$$E_{n*}(\Phi_{K(n)}(S^q))$$
 in another way?  
 $\Phi_{K(n)}(X) \xrightarrow{C^{S_K}} TAQ_{S_K}(S_K^{X_+}) \xleftarrow{model} \Delta^q.$  (10)



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### Rezk's approach

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- Calculate  $E_{n*}(\Phi_{K(n)}(S^q))$  in another way?  $\Phi_{K(n)}(X) \xrightarrow{C^{S_K}} TAQ_{S_K}(S_K^{X_+}) \xleftarrow{model} \Delta^q.$  (10)
- topological Andre-Quillen cohomology of the augmented commutative S<sub>K</sub>-algebra S<sup>X+</sup><sub>K</sub>.



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### Rezk's approach

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(10)

- topological Andre-Quillen cohomology of the augmented commutative S<sub>K</sub>-algebra S<sup>X+</sup><sub>K</sub>.
- $\Delta^q$ : Dyer-Lashof algebra.

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The E<sub>n\*</sub>-homology of the spectrum L(k)<sub>q</sub> is isomorphic to the dual of k<sup>th</sup> term of the Koszul resolution for Δ<sup>q</sup>.



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The E<sub>n\*</sub>-homology of the spectrum L(k)<sub>q</sub> is isomorphic to the dual of k<sup>th</sup> term of the Koszul resolution for Δ<sup>q</sup>.

• 
$$Ext^s_{\Delta^q}(\tilde{E}_{n,t}(S^q),\tilde{E}_{n,t}) \Rightarrow E_{n,q+t-s}(\Phi_{K(n)}(S^q)).$$



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### Summary of the above approach

#### This approach can be explained by the following diagram:

$$GT \xrightarrow{(-)^{GL_n(\mathbb{F}_p)}} H_c^*(G_n, E_{n*}(\Phi_{K(n)}(S^q))) \xrightarrow{(-)^{D^{\times}}} \pi_*(\Phi_{K(n)}(S^q)).$$
(11)



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### The problem we are facing

•  $\Phi_{K(n)}(S^q)$  is not a ring spectrum. We have few tools to deal with the differential in the  $E_2$ -page of the ANSS.



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The methods of calculation

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### The problem we are facing

- $\Phi_{K(n)}(S^q)$  is not a ring spectrum. We have few tools to deal with the differential in the  $E_2$ -page of the ANSS.
- Idea: swap the order of two SS?



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### New approach



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### Plan of work

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### • Understand the origin frame. (L(n), TAQ, $\Delta^q$ and so on.)



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### Plan of work

Zhonglin Wu

- Understand the origin frame. (L(n), TAQ,  $\Delta^q$  and so on.)
- Understand the duality in AG.



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### Plan of work

Zhonglin Wu

- Understand the origin frame. (L(n), TAQ,  $\Delta^q$  and so on.)
- Understand the duality in AG.
- Try to find these duality in the new frame.



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### Plan of work

Zhonglin Wu

- Understand the origin frame. (L(n), TAQ,  $\Delta^q$  and so on.)
- Understand the duality in AG.
- Try to find these duality in the new frame.
- Sample explicit calculation.



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## The End



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